A continuous auction model with insiders and random time of information release

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February 9, 2016

Abstract

In a unified framework we study equilibrium in the presence of an insider having information on the signal of the firm value, which is naturally connected to the fundamental price of the firm related asset. The fundamental value itself is announced at a future random (stopping) time. We consider two cases. First when the release time of information is known to the insider and then when it is unknown also to her. Allowing for very general dynamics, we study the structure of the insider's optimal strategies in equilibrium and we discuss market efficiency. In particular, we show that in the case the insider knows the information release time, the market is fully efficient. In the case the insider does not know this random time, we see that there is an equilibrium with no full efficiency, but where the sensitivity of prices is decreasing in time according with the probability that the announcement time is greater than the current time. In other words, the prices become more and more stable as the announcement approaches.

Key words: Market microstructure, equilibrium, insider trading, stochastic control, semimartingales, enlargement of filtrations.

JEL-Classification C61· D43· D44· D53· G11· G12· G14

MS-Classification 2010: 60G35, 62M20, 93E10, 94Axx

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The work of J. M. Corcuera is supported by the Spanish grant MTM2013-40782-P.
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1 Introduction

Models of financial markets with the presence of an insider or informational asymmetries have a large literature, see e.g. Karatzas and Pikovsky (1996), Amendiger et al. (1998), Grorud and Pointier (1998, 2001, 2005), Imkeller et al. (2001), Corcuera et al. (2004), Biagini and Øksendal (2005, 2006), Kohatsu-Higa (2007), Di Nunno et al. (2006, 2008), Biagini et al. (2012) and the references therein. In most of these models the prices are fixed exogenously, i.e. the insider does not affect the stock price dynamics, and the privileged information is a functional of the stock price process, e.g., the maximum, the final value, etc. As pointed by Danilova (2010), in an equilibrium situation market prices are determined by the demand of the market participants. So, in such a situation, the privileged information cannot be a functional of the stock price process because this implies the knowledge of the future demand and this is unrealistic. Then the privileged information is exogenous like the value of the fundamental price, or some signal of it, or the announcement time of the release of the fundamental price, which evolves independently of the demand. The questions considered in this paper deal with the existence of an equilibrium and the properties of the insider's optimal strategies. Moreover another question here studied is the efficiency of the market, namely the conditions under which the market prices converge to the fundamental one. These problems have been addressed in different works, with different degrees of generality, and with very different types of insider's privileged information and demands of the uninformed traders.

The original model is due to Kyle (1985), he considers three kinds of actors in the market: market makers, uninformed traders and one insider who knows the fundamental or liquidation value of an asset at certain fixed released time. In the model, there is also a price function establishing the relation between the market prices and the total demand. Kyle works in the discrete time setting and with noises given by Gaussian random walks. Back (1992) extends the previous work to the continuous time case. These are seminal papers which opened the way to various generalisations and extensions. To mention some here after, we have Back and Pedersen (1998), who consider a *dynamic* fundamental price and Gaussian noises with time varying volatility; Cho (2003), who considers pricing functions depending on the path of the demand process and studies what happens when the informed trader is risk-averse; Lasserre (2004), who considers a multivariate setting; Aase et al. (2012a), (2012b), who put emphasis in filtering techniques to solve the equilibrium problem; Campi and Çetin (2007), who consider a defaultable bond instead of a stock as in the Kyle-Back model and also consider the default time as privileged information; Danilova (2010), who deals with non-regular pricing rules; Caldentey and Stacchetti (2010) who take a random release time into account; and Campi et al. (2013), who consider again a defaultable bond, but this time the privileged information is represented by some dynamic signal related with the default time, see Example 28 below for more details.

The list could be completed with the references in the mentioned papers.

The present work extends the previous contributions in different ways. First we consider general noises for the demand processes, general pricing rules, random release times, and general dynamic information, all in the same model. Then, we study in detail the necessary conditions needed to have an equilibrium. These conditions are new in the literature. Specifically we consider the very general case in which an insider has access to some signal related to the firm fundamental value, which is in fact released at some stopping time. We first consider the case where the insider knows the random time of release of information and then the case where this is also unknown to her. We study these two situations in the same framework with the purpose of analyzing equilibrium and efficiency of the market.

Except for the multivariate setting of Lasserre (2004) and the insider risk-aversion attitude considered in some previous works, the framework we propose is one unique framework able to capture the diversified previous extensions of the Kyle-Back model mentioned above. We show this through various examples.

Our study shows explicitly to what extent *equilibrium* is a specific state of the market, a state which is induced by the interplay of the agents having different roles and asymmetric information. Indeed, the market makers set rational prices which are assumed to be a function H of the aggregate demand and time. For such H given, the insider optimizes her position to maximize her expected wealth. The necessary conditions for the existence of an equilibrium show how this optimization is possible only for some given pricing rules and under some available information flows.

In this study we show that the presence of the insider can be beneficial to the market from an efficiency point of view. In fact, if the insider knows the random release time, then the market is efficient. However, if this time of release is unknown even to the insider, then the market is *not* fully efficient. Nevertheless, an equilibrium can be reached if the sensitivity of the prices decreases in time according to the survival probability of the announcement. In other words, the prices become more stable as the announcement time approaches.

As far as we know this generality of the insider's information together with the presence of a random time of release has never been studied before. Moreover, the novelty of our contribution also concerns the use of very general dynamics for the demand process. In fact the insider's demand is allowed to be a general predictable semi-martingale. The present paper includes also various examples in which we give explicit insider's optimal strategies for a given pricing rule and define the concept of admissibility for pricing rules and insider strategies. Here we show how our results, coupled with the mathematical tools of enlargement of filtrations or filtering techniques, allow to finding explicitly the insider's optimal strategy in various cases

presented in the literature. We stress that our approach allows to treated all those cases in a unified framework.

The paper is structured as follows. In the next section we describe the model that gives rise to the stock prices. We discuss the insider's optimal strategies for a given pricing rule and define the concept of admissibility for pricing rules and insider strategies. In section 3 and 4 we discuss what happens when the release time is known to the insider or not, respectively. Finally, in section 5, we give some examples.

2 The model and equilibrium

We consider a market with two assets, a stock and a bank account with interest rate r equal to zero for the sake of simplicity. With abuse of terminology we will just write "prices" even though they are sometimes "discounted prices". The trading is continuous in time over the period $[0, \infty)$ and it is order driven. There is a (possibly random) release time $\tau < \infty$ a.s., when the fundamental value of the stock is revealed. The fundamental value process represents the actual value of the asset, which would be known only if *all* the information was public. The fundamental value is denoted by V and we shall give a precise definition later.

We shall denote the market price of the stock at time t by P_t. This represents the market evaluation of the asset. Just after the revelation time the price of the stock coincides with the fundamental value. Then we consider P_t defined only on $t \leq \tau$. It is possible that $P_t \neq V_t$ for $t \leq \tau$. We stop our studies at this (random) time of release τ.

We assume that all the random variables and processes mentioned are defined in the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that the filtrations are complete and right-continuous by taking, when necessary, their usual augmentation, as we shall specify below.

There are three kinds of traders. A *large* number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privileged information about the firm and can deduce its fundamental value, and the market makers, who set the market price and clear the market.

2.1 The agents and the equilibrium

At time t, let the insider information be given by $\mathcal{H}_t$ and denote her flow of information by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$. The informed trader, like any other trader, observes the market prices P and, in addition, she has access to the firm value, having actually access to some *signal process* η directly related to it. Moreover,

she will have *some* knowledge about the random time τ. In the sequel we will consider the two following cases:

- $\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t)$, i.e. the informed trader has knowledge of the time of release of information
- $\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t)$, i.e. the informed trader has no knowledge of this release time, but she will instantly know when it happens.

Here $\bar{\sigma}$ denotes the usual augmentation of a natural filtration σ (see [34], Ch. I, Def. 4.13). That is, e.g.,

$$\bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t) := \bigcap_{r>t} \left(\sigma(P_s, \eta_s, \tau, 0 \leq s \leq r) \cup \mathcal{N}\right),$$

where $\mathcal{N}$ is the family of $\mathbb{P}$-null sets in $\mathcal{F}$, and $\sigma(P_s, \eta_s, \tau, 0 \leq s \leq r)$ is the natural filtration generated by P, η, and τ.

In both the cases above, the insider has access to the fundamental value V which depends on the signal η. We do not specify any exact functional relationship between V and η, but we refer to the various examples in the sequel. In terms of the insider's information flow, we assume that V is a càdlàg $\mathbb{H}$-martingale (if not otherwise specified) such that $\sigma_V^2(t) := \frac{d[V,V]_t^c}{dt}$ is well defined (where $[V,V]^c$ indicates the continuous part of the quadratic variation of V).

Hereafter we describe in detail the three types of agents involved in this market model, namely their role, their demand process, and how they are perceived from the insider perspective. The assumptions now taken hold for any of the two insiders information flows described above and generically denoted by $\mathbb{H}$.

Let Z be the *aggregate* demand process of the **liquidity traders**. We recall that these are a large number of traders motivated by liquidity or hedging reasons. They are perceived as constituting noise in the market, thus also called *noise* traders. From the insider's perspective we assume that Z is a continuous $\mathbb{H}$-martingale, independent of η and V, such that $\sigma_Z^2(t) := \frac{\mathrm{d}[Z,Z]_t}{\mathrm{d}t}$ is well defined.

Market makers clear the market giving the market prices. They rely on the information given by the total aggregate demand Y which they observe. Specifically, $Y := X + Z$, where X denotes the insider demand process. X is naturally assumed $\mathbb{H}$-predictable process and it is also assumed to be a càdlàg $\mathbb{H}$-semimartingale. A motivation for this choice is given in the sequel. Just like the noise traders, the market makers instantly know about the time of release of information when that occurs. Hence, their information flow is: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \bar{\sigma}(Y_s, \tau \wedge s, 0 \leq s \leq t)$. From the economic point of view, due to the

competition among market makers, the market prices are *rational*, or *competitive*, in the sense that

$$P_t = \mathbb{E}(V_t|\mathcal{F}_t), \quad 0 \leq t \leq \tau. \tag{1}$$

This relationship gives the connection between the fundamental value of the firm and the market evaluation of the corresponding stock. In our model, consistent with the original idea of Kyle (1985) and later literature, we suppose that market makers give market prices through a pricing rule, which consists of a formula that here takes of the form:

$$P_t = H(t, \xi_t), \quad t \geq 0. \tag{2}$$

Here the deterministic function H is $C^{1,2}$ and $H(t, \cdot)$ is strictly increasing for all $t \geq 0$ and

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s,$$

where λ is a strictly positive deterministic function integrable with respect to Y. Observe that $\mathcal{F}_t = \bar{\sigma}(P_s, \tau \wedge s, 0 \leq s \leq t) \subseteq \mathcal{H}_t$, for all t and consequently P is an $\mathbb{F}$-martingale. Furthermore, from the assumptions on X and λ, we observe that ξ is a càdlàg $\mathbb{H}$-semimartingale. Hence, applying the Itô formula to (2), we can see that P is also an $\mathbb{H}$-semimartingale.

Definition 1 (Pricing rule) *Denote the class of such pairs (H, λ) above by $\mathfrak{H}$. An element of $\mathfrak{H}$ is called a pricing rule.*

The **informed trader** is assumed risk-neutral and she aims at maximizing her expected final wealth. Let W be the wealth process corresponding to insider's portfolio X. To illustrate the relationship among the processes V, P, X, and W we first consider a multi-period model where trades are made at times $i = 1, 2, \ldots N$, and where $\tau = N$ is random. If at time $i-1$, there is an order to buy $X_i - X_{i-1}$ shares, its *cost* will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total (cumulated) change at $\tau = N$ is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the convergence of the market and the fundamental prices at time $\tau = N$, there is the extra income: $X_N V_N$. So, the total wealth W_τ at τ is

$$\begin{aligned} W_\tau &= -\sum_{i=1}^{N} P_i (X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N. \end{aligned}$$

Consider now the continuous time setting where we have the processes X, P, and V, and we take N trading periods, where N is random and the trading times are: $0 \le t_1 \le t_2 \le ... \le t_N = \tau$, then we have

$$W_\tau = -\sum_{i=1}^{N} P_{t_{i-1}} (X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^{N} (P_{t_i} - P_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + X_{t_N} V_{t_N},$$

so, if the time between trades goes to zero, we will have

$$\begin{aligned} W_\tau &= X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau X_{t-} \mathrm{d}V_t + \int_0^\tau V_{t-} \mathrm{d}X_t + [V, X]_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-}) \, \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau, \end{aligned} \tag{3}$$

where (here and throughout the whole article) $P_{t-} = \lim_{s \uparrow t} P_s$ a.s. With the assumption taken on X to be an $\mathbb{H}$-*predictable* càdlàg semimartingale we can give meaning to the stochastic integrals above in the framework of Itô stochastic integration.

In the next subsection we discuss the characterization of an insider's optimal strategy in equilibrium in terms of fundamental value and insider information. Namely, we consider a process X that is optimal in the sense that it maximizes

$$J(X) := \mathbb{E}(W_\tau) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t, \xi_{t-})) \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau \right),$$

for a pricing rule $(H, \lambda) \in \mathfrak{H}$. However for technical and modelling reasons, we require additional properties to the triplet (H, λ, X).

Here and in the sequel $\partial_i H$, $\partial_{ij} H$ denote the first and second derivatives with respect to the i^{th}, i^{th} and j^{jh} variables, respectively.

Definition 2 (Admissibility) *We say that (H, λ, X) is an* admissible *triplet, if the process X (which may*

also be $X \equiv 0$) and price functions $(H,\lambda) \in \mathfrak{H}$ satisfy:

(A1) $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, *for all $t \geq 0$, where M is a continuous $\mathbb{H}$-martingale, A a bounded variation $\mathbb{H}$-predictable process, with $A_t = \sum_{0<s\leq t} (X_s - X_{s-})$, and θ a càdlàg $\mathbb{H}$-adapted process,*

(A2) $\mathbb{E}\left(\left(\int_0^\tau (\partial_2 H(s,\xi_s))^2 + (H(s,\xi_s))^2 + V_s^2\right)\left(\sigma_Z^2(s)\mathrm{d}s + \sigma_M^2(s)\mathrm{d}s\right)\right) < \infty$, *where* $\sigma_M^2(s) := \dfrac{\mathrm{d}[M,M]_s}{\mathrm{d}s}$,

(A3) $\mathbb{E}\left(\int_0^\tau (\partial_2 H(s,\xi_s) + H(s,\xi_s) + V_s)\,|\theta_s|\mathrm{d}s\right) < \infty$,

(A4) $\mathbb{E}\left(\sum_0^\tau \partial_2 H(s,\xi_{s-})|\Delta X_s|\right) < \infty, \Delta X_s := X_s - X_{s-}$,

(A5) $\mathbb{E}\left(\int_0^\tau \left(H^{-1}(\tau,\cdot)(V_{s-})\right)^2 + |Z_s|^2 + |X_{s-}|^2)\mathrm{d}[V,V]_s\right) < \infty$,

(A6) $\mathbb{E}\left(\int_0^\tau \lambda(s)|\partial_{22} H(s,\xi_s)|\left(\sigma_M^2(s) + |\sigma_{M,Z}(s)|\right))\mathrm{d}s\right) < \infty$,*where* $\sigma_{M,Z}(s) := \dfrac{\mathrm{d}[M,Z]_s}{\mathrm{d}s}$.

Remark 3 *Note that, since X is a càdlàg $\mathbb{H}$-*predictable *process, its martingale part cannot have jumps, see Corollary 2.31 in Jacod and Shiryaev (1987). Similarly, we have chosen Z to be a continuous $\mathbb{H}$-martingale before.*

Definition 4 (Optimality) *Let (H,λ,X) be an admissible triplet, the strategy X is called* optimal *with respect to a price process P if it maximizes $\mathbb{E}(W_\tau)$.*

Definition 5 (Equilibrium) *An admissible triplet (H,λ,X) is a (local) equilibrium, if the price process $P_\cdot := H(\cdot,\xi_\cdot)$ is rational* (1)*, given X, and the strategy X is (locally) optimal, given (H,λ).*

2.2 The optimality condition

In the sequel we will consider two kinds of stopping times: τ bounded, or τ finite but independent of (V,P,Z). In both cases, by the assumptions that V is an $\mathbb{H}$-martingale and X an $\mathbb{H}$-predictable càdlàg $\mathbb{H}$-semimartingale satisfying (A5), we have that $\mathbb{E}(\int_0^\tau X_t \mathrm{d}V_t) = 0$. In fact, we can argue that, if τ is bounded, we can apply Doob's Optional Sampling Theorem and, if τ is finite but independent of (V,P,Z) (and consequently of X), we have that

$$\mathbb{E}\left(\int_0^\tau X_t \mathrm{d}V_t\right) = \mathbb{E}\left(\mathbb{E}\left(\int_0^\tau X_s \mathrm{d}V_s \middle| \tau\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\int_0^t X_s \mathrm{d}V_s\right)\Big|_{t=\tau}\right) = 0.$$

Hence,

$$J(X) := \mathbb{E}\left(W_\tau\right) = \mathbb{E}\left(\int_0^\tau (V_{t-} - H(t,\xi_{t-}))\mathrm{d}X_t + [V,X]_\tau - [P,X]_\tau\right). \tag{4}$$

We now present a series of observations. First, note that

$$
\begin{aligned}
\int_0^{\tau}(V_{t-}-H(t,\xi_{t-}))\mathrm{d}X_t+[V,X]_\tau-[P,X]_\tau &= \int_0^{\tau-}(V_{t-}-H(t,\xi_{t-}))\mathrm{d}X_t+[V,X]_{\tau-}-[P,X]_{\tau-} \\
&\quad +(V_\tau-H(\tau,\xi_\tau))\,\Delta X_\tau .
\end{aligned}
$$

Then suppose that X is (locally) optimal and we modify only the last jump of this strategy by taking $(1+\varepsilon\gamma)\Delta X_\tau$, with γ an $\mathcal{H}_{\tau-}$-measurable and bounded random variable and $\varepsilon>0$ small enough. We recall that $\mathcal{H}_{\tau-} := \mathcal{H}_0 \vee \sigma(A\cap(\tau>t):\ A\in\mathcal{H}_t, t\geq 0)$ (see, e.g., Revuz and Yor (1999), page 46). Denote $X^{(\varepsilon)}$ this new strategy.

Then, since ΔX_τ is bounded (see (A1) in Definition 2), we can see that

$$
0=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,J(X^{(\varepsilon)})\Big|_{\varepsilon=0}=\mathbb{E}\left(\gamma\left((V_\tau-H(\tau,\xi_\tau))\,\Delta X_\tau-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2\right)\right),
$$

so we obtain

$$
\mathbb{E}\left((V_\tau-H(\tau,\xi_\tau))\,\Delta X_\tau-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\,(\Delta X_\tau)^2\Big|\,\mathcal{H}_{\tau-}\right)=0.
$$

Now we modify the strategy X by taking an $\mathbb{H}$-adapted càdlàg process β such that $X+\varepsilon\int\beta_s\mathrm{d}s$ is admissible, with $\varepsilon>0$ small enough.

We have

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}J(X+\varepsilon\int\beta_s\mathrm{d}s)\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left(\int_0^\tau(V_{t-}-H(t,\int_0^{t-}\lambda(s)(\mathrm{d}X_s+\varepsilon\beta_s\mathrm{d}s+\mathrm{d}Z_s)))(\mathrm{d}X_t+\varepsilon\beta_t\mathrm{d}t)\right)\Big|_{\varepsilon=0} \\
&- \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left([V,X+\varepsilon\int\beta_s\mathrm{d}s]_\tau-[H(\cdot,\int\lambda(s)(\mathrm{d}X_s+\varepsilon\beta_s\mathrm{d}s+\mathrm{d}Z_s),X+\varepsilon\int\beta_s\mathrm{d}s]_\tau\right)\Big|_{\varepsilon=0} \\
&= \mathbb{E}\left(\int_0^\tau(V_{t-}-H(t,\xi_t))\beta_t\mathrm{d}t\right)-\mathbb{E}\left(\int_0^\tau\partial_2H(t,\xi_{t-})\left(\int_0^t\lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}X_t\right) \\
&- \mathbb{E}\left(\left[\partial_2H(\cdot,\xi)\left(\int\lambda(s)\beta(s)\mathrm{d}s\right),X\right]_\tau\right) \\
&= \mathbb{E}\left(\int_0^\tau\left((V_t-H(t,\xi_t))-\lambda(t)\int_{t\wedge\tau}^\tau\partial_2H(s,\xi_{s-})\mathrm{d}X_s\right)\beta_t\mathrm{d}t\right) \\
&- \mathbb{E}\left(\int_0^\tau\left(\int_0^t\lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}\left[\partial_2H(\cdot,\xi),X\right]_t\right) \\
&= \mathbb{E}\left(\int_0^\tau\left((V_t-H(t,\xi_t))-\lambda(t)\left(\int_{t\wedge\tau}^\tau\partial_2H(s,\xi_{s-})\mathrm{d}X_s+[\partial_2H(\cdot,\xi),X]_t^\tau\right)\right)\beta_t\mathrm{d}t\right),
\end{aligned}
$$

where $[\cdot,\cdot]_t^\tau := [\cdot,\cdot]_\tau - [\cdot,\cdot]_t$. Since we can take $\beta_t=\alpha_u\mathbf{1}_{(u,u+h]}(t)$, with an $\mathcal{H}_u$-measurable and bounded α_u,

we have

$$\mathbb{E}\left(\int_u^{u+h}\Big[\mathbb{E}\big(\,\mathbf{1}_{[0,\tau]}(t)\,(V_t-H(t,\xi_t))\big|\,\mathcal{H}_t\big)-\lambda(t)\mathbb{E}\Big(\int_{t\wedge\tau}^{\tau}\partial_2 H(s,\xi_{s-})\mathrm{d}X_s+[\partial_2 H(\cdot,\xi),X]_{t\wedge\tau}^{\tau}\Big|\,\mathcal{H}_t\Big)\Big]\mathrm{d}t\bigg|\,\mathcal{H}_u\right)=0$$

and this means that the process $\Xi_t,\ t\geq 0$:

$$\Xi_t:=\int_0^t\Big[\mathbb{E}\big(\mathbf{1}_{[0,\tau]}V_u|\mathcal{H}_u\big)-\mathbb{E}\big(\mathbf{1}_{[0,\tau]}(u)H(u,\xi_u)|\mathcal{H}_u\big)-\lambda(u)\mathbb{E}\Big(\int_{u\wedge\tau}^{\tau}\partial_2 H(s,\xi_{s-})\mathrm{d}X_s+[\partial_2 H(\cdot,\xi),X]_{u\wedge\tau}^{\tau}\,|\mathcal{H}_u\Big)\Big]\mathrm{d}u$$

is an $\mathbb{H}$-martingale with bounded variation. In particular this implies that, for a.a. $t\geq 0$,

$$\mathbb{E}\left(\mathbf{1}_{[0,\tau]}(t)V_t\big|\,\mathcal{H}_t\right)-\mathbb{E}\left(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)\big|\,\mathcal{H}_t\right)-\lambda(t)\mathbb{E}\left(\int_{t\wedge\tau}^{\tau}\partial_2 H(s,\xi_{s-})\mathrm{d}X_s+[\partial_2 H(\cdot,\xi),X]_{t\wedge\tau}^{\tau}\,|\mathcal{H}_t\right)=0,a.s.$$

Since τ is an $\mathbb{H}$-stopping time, then for a.a. t and for a.a. $\omega\in\{\tau\geq t\}$, or equivalently a.s. on the stochastic interval $[[0,\tau]]$, we can write

$$V_t-H(t,\xi_t)-\lambda(t)\mathbb{E}\left(\int_t^{\tau}\partial_2 H(s,\xi_s)\mathrm{d}^-X_s\bigg|\,\mathcal{H}_t\right)=0,$$

where we have used a shorthand notation by means of d^-X_s as the *backward* integral in the sense of Revuz and Yor (1999) (see page 144), here extended to semimartingales with jumps. As a summary we have the following necessary condition, which is instrumental for identifying insider's optimal strategies.

Theorem 6 *An admissible triple (H,λ,X) such that X is locally optimal for the insider satisfies the equations:*

$$\mathbb{E}\Big(\left(V_\tau-H(\tau,\xi_\tau)\right)\Delta X_\tau-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\left(\Delta X_\tau\right)^2\Big|\,\mathcal{H}_{\tau-}\Big)=0. \tag{5}$$

$$V_t-H(t,\xi_t)-\lambda(t)\mathbb{E}\left(\int_t^{\tau}\partial_2 H(s,\xi_s)\mathrm{d}^-X_s\bigg|\,\mathcal{H}_t\right)=0. \tag{6}$$

a.s. on $[[0,\tau]]$.

In the sequel we study two different cases of knowledge of τ from the insider's perspective. First the case in which the insider knows the exact time τ of release of information about the firm value, then we study the case when the insider does not know τ.

3 Case when τ is known to the insider

In this section we consider the case when the insider knows the release time of information τ. Namely, let $\sigma(\tau)$ be the σ-algebra generated by τ, then $\sigma(\tau) \subseteq \mathcal{H}_0$. Hence, at any time t, the insider relies on the information given by:

$$\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau, 0 \leq s \leq t).$$

Moreover, we assume that τ is bounded, so the analysis here below is consistent with the one of the previous section. These are standing assumptions throughout this section.

3.1 Necessary conditions for the equilibrium

Our first observation is that optimal strategies lead the market price to the fundamental one, which means that the market is efficient. In fact we have the following proposition.

Proposition 7 *If (H, λ, X) is admissible with X locally optimal, then the optimal strategy X has no jump at τ and the market is efficient, i.e.*

$$V_{\tau-} = H(\tau, \xi_{\tau-}) = H(\tau, \xi_\tau) = P_\tau \qquad a.s.$$

Proof. By the assumptions (A1) and (A2) in Definition 2, equation (6) can be rewritten:

$$\begin{aligned}
&V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s)\mathrm{d}^- X_s \middle| \mathcal{H}_t\right) \\
&= V_t - H(t,\xi_t) - \lambda(t)\int_t^\tau \mathbb{E}(\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t)\mathrm{d}s \\
&- \lambda(t)\sum_t^\tau \mathbb{E}\left(\partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\int_t^\tau \lambda(s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s \middle| \mathcal{H}_t\right) \\
&= 0 \qquad \text{a.s. on } [[0,\tau]].
\end{aligned}$$

Now by assumption (A3) in Definition 2 and Corollary (2.4) in Revuz and Yor (1999), we have that

$$\lim_{t\uparrow\tau}\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s \middle| \mathcal{H}_t\right) = 0.$$

Analogously we also have that

$$\lim_{t\uparrow\tau}\lambda(t)\mathbb{E}\left(\int_t^\tau \lambda(s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\middle|\,\mathcal{H}_t\right)=0\quad\text{a.s.},$$

whereas

$$\lim_{t\uparrow\tau}\lambda(t)\sum_t^\tau \mathbb{E}\left(\partial_2 H(s,\xi_s)\Delta X_s|\,\mathcal{H}_t\right)=\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau.$$

Consequently

$$V_{\tau-}-H(\tau,\xi_{\tau-})-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau=0\quad\text{a.s.}\tag{7}$$

Now consider equation (5) and recall that $\mathcal{H}_0\subseteq\mathcal{H}_{\tau-}$. Since V is an $\mathbb{H}$-martingale and τ is $\mathcal{H}_0$-measurable, then $\mathbb{E}(V_\tau|\mathcal{H}_{\tau-})=V_{\tau-}$ (see Jacod and Shiryaev (1987), Lemma 2.27). Moreover, since X is $\mathbb{H}$-predictable, Z is continuous (and consequently ξ is $\mathbb{H}$-predictable) and τ is $\mathcal{H}_0$-measurable, we have

$$\mathbb{E}\left(\left(H(\tau,\xi_\tau)\right)\Delta X_\tau+\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\left(\Delta X_\tau\right)^2\middle|\,\mathcal{H}_{\tau-}\right)=H(\tau,\xi_\tau)\Delta X_\tau+\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\left(\Delta X_\tau\right)^2.$$

Therefore equation (5) gives

$$\left(V_{\tau-}-H(\tau,\xi_\tau)\right)\Delta X_\tau-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\left(\Delta X_\tau\right)^2=0\quad\text{a.s.}\tag{8}$$

If it was $\Delta X_\tau\neq 0$, then we would have that

$$V_{\tau-}-H(\tau,\xi_\tau)-\lambda(\tau)\partial_2 H(\tau,\xi_\tau)\Delta X_\tau=0.$$

However, comparing the above equation with (7) we have that $H(\tau,\xi_\tau)=H(\tau,\xi_{\tau-})$, which actually contradicts $\Delta X_\tau\neq 0$, being H strictly increasing in the second variable. Then this shows that a (locally) optimal strategy X has no jump at τ and that $V_{\tau-}=H(\tau,\xi_{\tau-})=H(\tau,\xi_\tau)$, see by (7). ■

Remark 8 *In Aase et al. (2012a) it was already observed that market efficiency is a consequence of the optimality of the insider's strategy. Here we obtain an extension of this result for a more general behaviour of the fundamental value and of the demand process of the noise traders.*

Remark 9 *This efficiency situation is also the case in Campi and Çetin (2007). In our notation they have the signal $\eta=\bar{\tau}$, with $\bar{\tau}$ known by the insider and representing the default time of a bond with face value 1, the fundamental value $V_t=\mathbf{1}_{\{\bar{\tau}>1\}}$, and the release time is $\tau=\bar{\tau}\wedge 1$. So, τ is $\mathcal{H}_0$-measurable and it is*

bounded. Then, they obtain

$$\mathbf{1}_{\{\bar{\tau}>1\}} - H(\bar{\tau}\wedge 1, \xi_{\bar{\tau}\wedge 1}) = 0 \quad a.s.$$

Within this study, the authors also assume that $\bar{\tau}$ is the first passage time of a standard Brownian motion independent of Z.

Remark 10 *If we take the fundamental value $V_t \equiv V$ and the deterministic fixed release time $\tau \equiv 1$, then we retrive in Back's framework (1992). There it is shown that market prices converge to V when $t \to 1$.*

Hereafter we consider necessary conditions for the existence of an equilibrium. These conditions show the synergy between the optimal insider strategy and the pricing rule in an equilibrium state. Note that one cannot use these conditions to (uniquely) identify a pricing rule. The choice of pricing rules is not unique. In the next subsection we will study necessary and sufficient conditions for the existence of an equilibrium for a wide class of pricing rules. Before that we have the following result.

Proposition 11 *Consider an admissible triple (H, λ, X), with $\lambda \in C^1$. If (H, λ, X) is a local equilibrium, we have:*

(i) $H(\tau, \xi_\tau) = V_{\tau-} \quad a.s.,$

(ii) $\dfrac{\lambda'(t)}{\lambda^2(t)}V_t - \dfrac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \dfrac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \dfrac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\big(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)\big) = 0 \quad a.s.\ on\ [[0,\tau))$

(iii) $\partial_1 H(t,\xi_t) + \dfrac{1}{2}\partial_{22}H(t,\xi_t)\lambda^2(t)\mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t)|\mathcal{F}_t\right) = 0 \quad a.s.\ on\ [[0,\tau)).$

Proof. (i) It is just Proposition 7. We prove (ii) and (iii). By using Itô formula on $\frac{H(t,\xi_t)}{\lambda(t)}$, with (A2) in Definition 2 applied, we have

$$\begin{aligned}
\mathbb{E}\left(\int_t^\tau \frac{1}{\lambda(s)}\partial_2 H(s,\xi_{s-})\mathrm{d}\xi_s \,|\mathcal{H}_t\right) &= \mathbb{E}\left(\frac{H(\tau,\xi_\tau)}{\lambda(\tau)}\bigg|\,\mathcal{H}_t\right) - \frac{H(t,\xi_t)}{\lambda(t)} \\
&\quad - \mathbb{E}\left(\int_t^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_Y^2(s)\right)\mathrm{d}s\bigg|\,\mathcal{H}_t\right) \\
&\quad - \mathbb{E}\left(\sum_{t\le s\le\tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_{s-})\Delta X_s\right)\bigg|\,\mathcal{H}_t\right),
\end{aligned}$$

where $\sigma_Y^2(s) := \frac{\mathrm{d}[Y,Y]^c_s}{\mathrm{d}s}$. Since X is locally optimal given (H, λ), by the equation (6) and the Proposition 7 we can write:

$$
\begin{aligned}
0 &= V_t - \lambda(t)\mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)} \middle| \mathcal{H}_t \right) \\
&+ \lambda(t) \int_t^\tau \mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_Y^2(s) \middle| \mathcal{H}_t \right) \mathrm{d}s \\
&+ \lambda(t) \sum_{t \le s \le \tau} \mathbb{E}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t \right) \\
&- \lambda(t) \int_t^\tau \mathbb{E}\left(\lambda(s)\partial_{22}H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s)) | \mathcal{H}_t\right) \mathrm{d}s.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
0 &= \frac{V_t}{\lambda(t)} - \frac{V_t}{\lambda(\tau)} \\
&+ \int_t^\tau \mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \middle| \mathcal{H}_t \right) \mathrm{d}s \\
&+ \sum_{t \le s \le \tau} \mathbb{E}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t \right),
\end{aligned}
\tag{9}
$$

where $\sigma_{M,Y}(t) := \frac{\mathrm{d}[M,Y]_t}{\mathrm{d}t} = \sigma_M^2(t) + \sigma_{M,Z}(t)$. We study the summands in the previous expression. By taking infinitesimal increments over time, we can identify the predictive and martingale parts. In fact, for the first term we have

$$
\mathrm{d}\left(\frac{V_t}{\lambda(t)} - \frac{V_t}{\lambda(\tau)} \right) = -\frac{\lambda'(t)}{\lambda^2(t)} V_t \mathrm{d}t + \left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(\tau)} \right) \mathrm{d}V_t.
$$

If we define,

$$
\mathcal{M}_t := \mathbb{E}\left(\int_0^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \right) \mathrm{d}s \middle| \mathcal{H}_t \right), \quad t \ge 0,
$$

we can see that $\mathcal{M}$ is an $\mathbb{H}$-martingale and, since τ is $\mathcal{H}_0$-measurable, we have

$$
\begin{aligned}
&\mathrm{d} \int_t^\tau \mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)(\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \middle| \mathcal{H}_t \right) \mathrm{d}s \\
&= \left(\frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) - \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} - \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \right) \mathrm{d}t + \mathrm{d}\mathcal{M}_t
\end{aligned}
$$

for the second term. Analogously the for the third summand we have

$$
\mathrm{d} \sum_{t \le s \le \tau} \mathbb{E}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t \right) = -\frac{\Delta H(t,\xi_t) - \partial_2 H(t,\xi_t)\Delta\xi_t}{\lambda(t)} + \mathrm{d}\mathcal{L}_t,
$$

with

$$\mathcal{L}_t := \mathbb{E}\left(\sum_{0\leq s\leq \tau} \frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t\right).$$

Therefore the sums of the predictive parts gives:

$$\begin{aligned} 0 = &\frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \\ &+ \frac{\Delta H(t,\xi_t) - \partial_2 H(t,\xi_t)\Delta\xi_t}{\lambda(t)} \quad \text{a.s. on } [[0,\tau)) \ . \end{aligned} \tag{10}$$

Then the continuous and jump parts of the right-hand side of the previous equation will be equal to zero. So

$$\frac{\Delta H(t,\xi_t) - \partial_2 H(t,\xi_t)\Delta\xi_t}{\lambda(t)} = 0 \quad \text{a.s. on } [[0,\tau)) \tag{11}$$

and

$$0 = \frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \quad \text{a.s. on } [[0,\tau)), \tag{12}$$

which gives (ii). Recall that (X,λ,H) is a local equilibrium and that the prices are rational given X. So, by taking conditional expectations with respect to $\mathcal{F}_t$ in (12), we have

$$\begin{aligned} 0 &= \frac{\lambda'(t)}{\lambda^2(t)}\left(\mathbb{E}(V_t|\mathcal{F}_t) - \mathbb{E}(H(t,\xi_t)|\mathcal{F}_t)\right) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\mathbb{E}\left(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)|\mathcal{F}_t\right) \\ &= \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\left(\sigma_Y^2(t) - 2\mathbb{E}\left(\sigma_{M,Y}(t)|\mathcal{F}_t\right)\right) \quad \text{a.s. on } [[0,\tau)), \end{aligned} \tag{13}$$

which gives (iii). ■

Proposition 12 *Assume that (X,λ,H) with $\lambda \in C^1$ is a local equilibrium. If in addition the pricing rule $H(t,\cdot)$ is linear, for all t, or the optimal strategy X is absolutely continuous, then we have:*
(i) Y is an $\mathbb{F}$-local martingale;
(ii) If $V_t \neq P_t$ a.s. on $[[0,\tau))$, then $\lambda(t) = \lambda_0 > 0$.

Proof. (i) From (11) and (13) we have

$$\mathrm{d}P_t = dH(t,\xi_t) = \lambda(t)\partial_2 H(t,\xi_{t-})\mathrm{d}Y_t,$$

and, since P is an $\mathbb{F}$- martingale and $\lambda(t)\partial_2 H(t,y) > 0$, we have that Y is an $\mathbb{F}$- local martingale.

(ii) From (11) and (13) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) = 0,$$

then $V_t \neq H(t,\xi_t)$ implies that $\lambda'(t) = 0$. ■

Remark 13 *We shall see later that if the pricing rule is linear then the optimal strategy is in fact absolutely continuous.*

Example 14 *Consider the case* $\tau \equiv 1$, $V_t \equiv V$ *and such that* $\log V \sim N(m, v^2)$, $Z = \sigma B$ *where* B *is a Brownian motion. Assume that the price functions are of the form*

$$H(t,u) = \exp\left\{ m + \frac{v^2}{2} + \frac{v}{\lambda}\frac{1}{\sigma(1-\alpha)}u - \frac{1}{2}\frac{1+\alpha}{1-\alpha}v^2 t \right\}, \quad 0 < \alpha < 1.$$

Note that

$$\partial_1 H(t,u) = -\frac{1}{2}H(t,u)\frac{1+\alpha}{1-\alpha}v^2$$

and

$$\partial_{22} H(t,u) = H(t,u)\left(\frac{v}{\lambda}\right)^2 \frac{1}{\sigma^2(1-\alpha)^2}.$$

So we have

$$\partial_1 H(t,u) + \frac{1}{2}\partial_{22}H(t,u)\lambda^2\sigma^2(1-\alpha^2) = 0.$$

We look for optimal strategies of the form

$$\mathrm{d}X_t = \mathrm{d}M_t + \mathrm{d}\theta_t,$$

where M *is an* $\mathbb{F}$*-martingale and such that* $[X,Z]_t = -\alpha\sigma^2 t$, $0 \le t \le 1$. *Let* $\bar{Y}$ *be the solution of*

$$\bar{Y}_t = \sigma(1-\alpha)B_t + \int_0^t \frac{\bar{Y}_1 - \bar{Y}_s}{1-s}\mathrm{d}s,$$

where we take

$$\bar{Y}_1 = \sigma(1-\alpha)\frac{\log V - m}{v}.$$

Then if we take

$$X_t = -\sigma\alpha B_t + \int_0^t \frac{\bar{Y}_1 - \bar{Y}_s}{1-s}\mathrm{d}s + v\sigma\alpha t, \quad 0 \le t \le 1,$$

we also have that

$$Y_1 = \bar{Y}_1 + v\sigma\alpha,$$

and

$$P_1 = H(1, \lambda Y_1) = \exp\left\{m + \frac{v}{\sigma(1-\alpha)}\bar{Y}_1\right\} = V.$$

Then X satisfies the necessary conditions to be an equilibrium. We shall see that X satisfies also the sufficient conditions, by the Theorem 15 in the next section.

3.2 Characterization of the equilibrium

In this subsection we shall give necessary and sufficient conditions to guarantee that (H, λ, X) is an equilibrium in the context of pricing rules $(H, \lambda) \in \mathfrak{H}$ satisfying

$$0 = \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\lambda(t)^2\sigma^2(t) \qquad \text{a.a. } t \geq 0,\ y \in \mathbb{R}, \tag{14}$$

where σ^2 is a deterministic and càdlàg function and $0 < \sigma^2(t) \leq \sigma_Z^2(t)$ for a.a. t. Condition (14) specifies a subclass of pricing rules (Definition 1) and thus of admissible strategies (Definition 2). Note that condition (14) is close to condition (ii) in Proposition 11 (with $\sigma^2(t) = \sigma_Z^2(t) - \sigma_M^2(t)$), which is a necessary condition for the equilibrium. Observe also that the pricing rules (H, λ) are, by construction, deterministic and consequently, except for linear pricing rules, we need $\sigma_Z^2(t) - \sigma_M^2(t)$ to be deterministic in equilibrium.

For the sake of simplicity, in this subsection, we shall assume that V is continuous. Recall that the release time τ is known to the insider. We have the following necessary and sufficient conditions for the equilibrium.

Theorem 15 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (14) with $\partial_{22} H(t, y) > 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$, $\lambda(t) = \lambda_0 > 0$, and $\int_0^t \mathbb{E}\Big(\big(\partial_2 H(s, \lambda_0 \int_0^t \sigma(s)\mathrm{d}B_s)\big)^2\Big)\sigma^2(s)\mathrm{d}s < \infty$, for all $t \geq 0$, where B is a Brownian motion independent of τ. Assume that the fundamental value V is a continuous $\mathbb{H}$-martingale. Then an equilibrium exists if and only if the following conditions hold:*

(i) $H(\tau, \xi_\tau) = V_\tau$

(ii) $\sigma_M^2(t) = \sigma_Z^2(t) - \sigma^2(t), \quad 0 \leq t < ess\sup\tau$

(iii) $Y \ = X + Z$ *has no jumps*

(iv) $Y_t + \lambda_0 \displaystyle\int_0^t \frac{\partial_{22} H(s, \xi_s)}{\partial_2 H(s, \xi_s)}(\sigma_{M,Z}(s) + \sigma_M^2(s))\mathrm{d}s,\ 0 \leq t < ess\sup\tau,$ *is an $\mathbb{F}$-local martingale.*

Proof. Assume $(i)-(iv)$, we show that (H,λ,X) is an equilibrium. Consider a process ς such that

$$\varsigma_t := \lambda_0 \int_0^t \sigma(s)\mathrm{d}B_s.$$

where B is a Brownian motion independent of τ. First if $H(t,y)$ is a solution of (14).

$$H(t,\varsigma_t) = H(0,0) + \lambda_0 \int_0^t \partial_2 H(s,\varsigma_s)\sigma(s)\mathrm{d}B_s,$$

then, by the hypothesis, $(H(t,\varsigma_t))_{t\geq 0}$ is a martingale (w.r.t. its own filtration) and since ς has independent increments and τ is bounded and independent of ς

$$H(t\wedge\tau,y) = \mathbb{E}\left(H(\tau,\varsigma_\tau)\right|\varsigma_{t\wedge\tau}=y,\tau) = \mathbb{E}\left(H(\tau,y+\varsigma_\tau-\varsigma_{t\wedge\tau})\right|\tau).$$

Set now, for $T\in[0,\infty)$,

$$i(T,y,v) := \int_y^{H^{-1}(T,\cdot)(v)} \frac{v-H(T,x)}{\lambda_0}\mathrm{d}x$$

and define

$$I(t,y,v) := \mathbb{E}\left(i(\tau,y+\varsigma_\tau-\varsigma_{t\wedge\tau},v)\right|\tau), \quad t\geq 0.$$

Note that $I(t,y,v)$ is a random-field. We have that

$$\begin{aligned}\partial_2 I(t,y,v) &= \mathbb{E}\left(\partial_1 i(\tau,y+\varsigma_\tau-\varsigma_{t\wedge\tau},v)\right|\tau)\\ &= \mathbb{E}\left(-\frac{v+H(\tau,y+\varsigma_\tau-\varsigma_{t\wedge\tau})}{\lambda_0}\middle|\tau\right) = \frac{-v+H(t\wedge\tau,y)}{\lambda_0}.\end{aligned} \tag{15}$$

We can take the derivative under the integral sign because $H(\tau(\omega),\cdot)$ is monotone and $\mathbb{E}(H(\tau,\varsigma_\tau)|\tau)<\infty$. Then $I(t,y,v)$ is well defined and

$$\begin{aligned}I(t,y,v) &= \mathbb{E}\left(i(\tau,y+\varsigma_\tau-\varsigma_{t\wedge\tau},v)\right|\tau)\\ &= \mathbb{E}(\,i(\tau,\varsigma_\tau,v)|\,\varsigma_{t\wedge\tau}=y,\tau),\end{aligned}$$

then, fixed v, $(I(t,\varsigma_{t\wedge\tau},v))_{t\geq 0}$ is a martingale (w.r.t. its own filtration), so

$$\partial_1 I(t,\varsigma_t,v) + \frac{1}{2}\partial_{22} I(t,\varsigma_t,v)\lambda_0^2\sigma^2(t) = 0, \; a.s. \text{ on } [[0,\tau]]. \tag{16}$$

Now, consider an admissible strategy X, by using Itô-Wentzell's formula, we have

$$
\begin{aligned}
I(\tau, \xi_\tau, V_\tau) = I(0,0,V_0) &+ \int_0^\tau \partial_3 I(t, \xi_t, V_t)\mathrm{d}V_t + \int_0^\tau \partial_1 I(t, \xi_t, V_t)\mathrm{d}t \\
&+ \int_0^\tau \partial_2 I(t, \xi_{t-}, V_t)\mathrm{d}\xi_t + \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\mathrm{d}[\xi^c, \xi^c]_t \\
&+ \int_0^\tau \partial_{23} I(t, \xi_t, V_t)\mathrm{d}[\xi^c, V]_t + \frac{1}{2}\int_0^\tau \partial_{33} I(t, \xi_t, V_t)\sigma_V^2(t)\mathrm{d}t \\
&+ \sum_{0\le t\le\tau} \left(\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t)\Delta\xi_t\right).
\end{aligned}
$$

By construction, $\xi_0 = 0$, by (i) $\mathrm{d}\xi_t = \lambda_0 \mathrm{d}Y_t$. Now we have that

$$
\mathrm{d}[\xi^c, \xi^c]_t = \lambda_0^2 \mathrm{d}[X^c, X^c]_t + 2\lambda_0^2 \mathrm{d}[X^c, Z]_t + \lambda_0^2 \sigma_Z^2(t)\mathrm{d}t.
$$

Also by (15) and the fact that V and Z are independent,

$$
\partial_{23} I(t, \xi_t, V_t)\mathrm{d}[\xi^c, V]_t = -\frac{1}{\lambda_0}\mathrm{d}[\xi^c, V]_t = -\mathrm{d}[X, V]_t,
$$

then using (15) and (16), and the fact that Z has not jumps, we get

$$
\begin{aligned}
I(\tau, \xi_\tau, V_\tau) = I(0,0,V_0) &+ \int_0^\tau \partial_3 I(t, \xi_t, V_t)\mathrm{d}V_t + \int_0^\tau (P_{t-} - V_t)(\mathrm{d}X_t + \mathrm{d}Z_t) \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda(t)^2 \mathrm{d}[X^c, X^c]_t - [X, V]_\tau + \frac{1}{2}\int_0^\tau \partial_{33} I(t, \xi_t, V_t)\sigma_V^2(t)\mathrm{d}t \\
&+ \int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda(t)^2 \mathrm{d}[X^c, Z] + \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda(t)^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0\le t\le\tau} \left(\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t)\lambda(t)\Delta X_t\right).
\end{aligned}
$$

Subtracting $[P, X]_\tau$ from both sides and rearranging the terms, we obtain

$$
\begin{aligned}
&\int_0^\tau (V_t - P_{t-})\mathrm{d}X_t - [P, X]_\tau + [X, V]_\tau - \left(I(0,0,V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t, \xi_t, V_t)\sigma_V^2(t)\mathrm{d}t\right) \\
&= -I(\tau, \xi_\tau, V_\tau) + \int_0^\tau \partial_3 I(t, \xi_{t-}, V_t)\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}[X^c, X^c]_t + \int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \mathrm{d}[X^c, Z]_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t, \xi_t, V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0\le t\le\tau} \left(\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t)\lambda_0 \Delta X_t\right) - [P, X]_\tau. \qquad (17)
\end{aligned}
$$

We have that

$$[P,X]_\tau = [P^c,X^c]_\tau + \sum_{0\le t\le\tau} \Delta P_t \Delta X_t.$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c$, so by using (15), we obtain

$$\begin{aligned}
[P^c,X^c]_\tau &= \left[\int_0^\cdot \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c, X^c\right]_\tau = \int_0^\tau \partial_2 H(t,\xi_t)\mathrm{d}\left[\xi^c,X^c\right]_t \\
&= \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t + \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \mathrm{d}\left[X^c,Z\right]_t,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_0\partial_2 I(t,\xi_{t-},V_t)\Delta X_t + \Delta P_t\Delta X_t &= (P_{t-}-V_t)\Delta X_t + \Delta P_t \Delta X_t \\
&= (P_t - V_t)\Delta X_t = \lambda_0 \partial_2 I(t,\xi_t,V_t)\Delta X_t.
\end{aligned}$$

Substituting the above relationships in the right-hand side of the equation (17), it becomes

$$\begin{aligned}
&-I(\tau,\xi_\tau,V_\tau) + \int_0^\tau \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t - \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \mathrm{d}[X^c,X^c]_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2\left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0\le t\le\tau}\left(I(t,\xi_t,V_t) - I(t,\xi_{t-},V_t) - \lambda_0\partial_2 I(t,\xi_t,V_t)\Delta X_t\right) \\
&= -I(\tau,\xi_\tau,V_\tau) + \int_0^\tau \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t \\
&+ \sum_{0\le t\le\tau}\left(I(t,\xi_t,V_t) - I(t,\xi_{t-},V_t) - \lambda_0\partial_2 I(t,\xi_t,V_t)\Delta X_t\right).
\end{aligned} \tag{18}$$

Recall the expected total wealth of an insider's strategy (4). Then, taking the expectation in the right-hand side of (17), or equivalently of (18), we show that the maximum is achieved at X. For this it is important to note that $\partial_{33} I(t,y,v)$ does not depend on y and so $\partial_{33} I(t,\xi_t,V_t)$ does not depend of ξ. Then $I(0,0,V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t$ has the same value for *any* insider's strategy. The result follows from the following points.

1. (iii) guarantees that $\Delta X_t = 0$

2. The processes $\int_0^\cdot \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t$ and $\int_0^\cdot (P_t - V_t)\mathrm{d}Z_t$ are $\mathbb{H}$-martingales by (A5) and (A2) in Definition 2, hence they have null expectation

3. We know that $\lambda_0\partial_{22}I(\tau,\xi_\tau,V_\tau) = \partial_2 H(\tau,\xi_\tau) > 0$ and that $\lambda_0\partial_2 I(\tau,\xi_\tau,V_\tau) = -V_\tau + H(\tau,\xi_\tau)$ so by (i) we have a maximum value of $-E[I(\tau,\xi_\tau,V_\tau)]$ for our strategy X.

Assumption (iv) and (i) together with condition (A2) in Definition 2 guarantee the rationality of prices, given X. In fact from (14)

$$\mathrm{d}P_t = \lambda_0\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \frac{1}{2}\lambda_0^2\left(\sigma_Y^2(t) - \sigma^2(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t$$

and by (ii)

$$\begin{aligned}\mathrm{d}P_t &= \lambda_0\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \lambda_0^2\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t\\ &= \lambda_0\partial_2 H(t,\xi_t)\left(\mathrm{d}Y_t + \lambda_0\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\frac{\partial_{22}H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t\right)\end{aligned}$$

so, P is an $\mathbb{F}$-local martingale and, by condition (A2) in Definition 2, it is an $\mathbb{F}$-martingale. Then from (i), and on the set $\{t \le \tau\}$ we have

$$\mathbb{E}\left(H(\tau,\xi_\tau)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_\tau\middle|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(V_\tau\middle|\mathcal{H}_t\right)\middle|\mathcal{F}_t\right) = \mathbb{E}\left(V_t\middle|\mathcal{F}_t\right).$$

Conversely, assume that (H,λ,X) is an equilibrium. We show that $(i)-(iv)$ hold true. First note that (i) is a necessary condition for equilibrium by (i) in Proposition 11. Also (ii) is a necessary condition, in fact by (ii) in Proposition 11 and since $\lambda(t) = \lambda_0$ we obtain that, in the equilibrium,

$$\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda_0^2(\sigma_Z^2(t) - \sigma_M^2(t)) = 0,$$

now since $\partial_{22}H(t,y) > 0$ and (14) holds, we have that $\sigma_Z^2(t) - \sigma_M^2(t) = \sigma^2(t)$. From the computations above we can see that $\partial_{22}I = \frac{\partial_2 H}{\lambda_0} > 0$ (convexity) implies that

$$I(t,x+h,v) - I(t,x,v) - \partial_2 I(t,x,v)h \le 0, \quad \text{for any } h.$$

So,

$$\sum_{0\le t\le\tau}\left(I(t,\xi_{t-} + \lambda_0\Delta X_t, V_t) - I(t,\xi_{t-},V_t) - \partial_2 I(t,\xi_t,V_t)\lambda_0\Delta X_t\right) \le 0.$$

Since X is optimal, then $\Delta X_t = 0$. So (iii) is a necessary condition for equilibrium. Finally, from the Itô

formula, we have that

$$\mathrm{d}Y_t + \lambda_0 \left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right) \frac{\partial_{22} H(t,\xi_t)}{\partial_2 H(t,\xi_t)} \mathrm{d}t = \frac{\mathrm{d}P_t}{\lambda_0 \partial_2 H(t,\xi_t)}.$$

Since prices are rational, given X, then we see that (iv) holds true. ■

For the linear pricing rules case we have the following characterization.

Theorem 16 *Consider an admissible triple (H,λ,X) with (H,λ) satisfying (14) with $\partial_{22}H(t,y) = 0$ for all $(t,y) \in \mathbb{R}_+ \times \mathbb{R}$ and $\lambda(t) = \lambda_0$ for $t \geq 0$. Assume that V is a continuous $\mathbb{H}$-martingale. Then we have an equilibrium if and only if the following conditions hold:*

(i) $H(\tau, \xi_\tau) = V_\tau$

(ii) $\sigma_M(t) = 0$, $\quad 0 \leq t < ess \sup \tau$

(iii) $Y = X + Z$ *has no jumps*

(iv) $Y_t,\ 0 \leq t \leq ess \sup \tau,$ *is an $\mathbb{F}$-local martingale*

Proof. Let (H,λ,X) be an equilibrium. First note that if H is linear and $\lambda(t) = \lambda_0$, point (ii) in Proposition 11 does not imply any condition on σ_M. Secondly,

$$\mathrm{d}P_t = \lambda_0 \partial_2 H(t,\xi_t) \mathrm{d}Y_t,$$

so we have (iv). Thirdly, by (15), $\partial_{22} I(t,\xi_t,V_t) = C(t) > 0$ (deterministic). So, in the linear case, the term

$$\frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right) \mathrm{d}t,$$

in the equality (17), does not depend on the insider's strategy. Then we can pass the term to the left-hand side of (17) and $I(0,0,V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2 \mathrm{d}t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right) \mathrm{d}t$ becomes a true

bound for the insider's wealth. We have that

$$
\begin{aligned}
&\int_0^\tau (V_t - P_{t-})\mathrm{d}X_t - [P,X]_\tau + [X,V]_\tau \\
&- \left(I(0,0,V_0) + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2 \mathrm{d}t + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \left(\sigma_Z^2(t) - \sigma^2(t)\right)\mathrm{d}t \right) \\
&= -I(\tau,\xi_\tau,V_\tau) + \int_0^\tau \partial_3 I(t,\xi_{t-},V_t)\mathrm{d}V_t + \int_0^\tau (P_t - V_t)\mathrm{d}Z_t \\
&- \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \mathrm{d}[X^c,X^c]_t + \sum_{0\le t\le \tau} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_t,V_t)\lambda_0 \Delta X_t\right).
\end{aligned}
$$

Now, the arguments in the proof of the previous theorem apply and we obtain (i) and (iii) . Finally, since

$$
-\frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda_0^2 \mathrm{d}[X^c,X^c]_t \le 0,
$$

its maximum value is achieved if and only if $[X^c,X^c]\equiv 0$ and we conclude (ii).

The converse is directly obtained from the previous theorem. ■

4 Case when τ is unknown to the insider

In this section we consider the case when the insider does not know the precise time τ of release of information. Namely, the insider's information flow is given by:

$$
\mathcal{H}_t = \bar{\sigma}(P_s, \eta_s, \tau \wedge s, 0 \le s \le t).
$$

Moreover we assume that τ finite is independent of (V,P,Z) , that $\mathbb{P}(\tau > t) > 0$ for all $0 \le t < T \in \bar{\mathbb{R}}_+$ and that τ has a density with respect to the Lebesgue measure. For the sake of simplicity we also assume that V is a continuous $\mathbb{H}$-martingale. These are standing assumptions for this section.

Proposition 17 *Consider an admissible triple (H,λ,X) with $\lambda \in C^1$ and $\lim_{\bar{T}\uparrow T} \frac{\mathbb{P}(\tau>\bar{T})}{\lambda(\bar{T})} =: c < \infty$. If (H,λ,X) is a local equilibrium, we have:*

(i) $\lim_{\bar{T}\uparrow T} H(\bar{T},\xi_{\bar{T}}) = \lim_{\bar{T}\uparrow T} V_{\bar{T}}$ *a.s.*

(ii) $\partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)}\right)(V_t - H(t,\xi_t)) + \frac{\mathbb{P}(\tau > t)}{\lambda(t)}\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22}H(t,\xi_t)\mathbb{P}(\tau > t)\lambda(t)(\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) = 0,$

(iii) $\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda^2(t)\mathbb{E}\left(\sigma_Z^2(t) - \sigma_M^2(t)|\mathcal{F}_t\right) = 0$ *a.s. on* $[[0,\tau]]$,

Proof. Going back to Theorem 6, we can see that equation (6) can be written as:

$$V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}(\int_t^T \mathbf{1}_{[0,\tau]}(s)\left(\partial_2 H(s,\xi_s)\mathrm{d}^- X_s\right)|\mathcal{H}_t) = 0 \quad \text{a.s. on } [[0,\tau]].$$

We recall that the optimal total demand X for the insider satisfies (A1) - (A6) in Definition 2. Then we have

$$\begin{aligned}
0 &= V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\left(\partial_2 H(s,\xi_s)\mathrm{d}^- X_s\right)\middle| \mathcal{H}_t\right) \\
&= V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\sum_t^T \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\Delta X_s\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\int_t^T \mathbf{1}_{[0,\tau]}(s)\lambda(s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\middle| \mathcal{H}_t\right) \\
&= V_t - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s|\mathcal{H}_t)\left(\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\right)\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\sum_t^T \mathbb{P}(\tau > s|\mathcal{H}_t)\partial_2 H(s,\xi_s)\Delta X_s\middle| \mathcal{H}_t\right) \\
&- \lambda(t)\mathbb{E}\left(\mathbb{P}(\tau > s|\mathcal{H}_t)\lambda(s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\middle| \mathcal{H}_t\right).
\end{aligned}$$

Observe that, on $t < \tau \le T$ and for $s > t$,

$$\mathbb{P}(\tau > s|\mathcal{H}_t) = \mathbb{P}(\tau > s|\tau > t) = \frac{\mathbb{P}(\tau > s)}{\mathbb{P}(\tau > t)}.$$

Hence, substituting in the previous espression, we have

$$\begin{aligned}
0 = V_t - H(t,\xi_t) &- \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s)\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\middle| \mathcal{H}_t\right) \\
&- \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}\left(\sum_t^\infty \mathbb{P}(\tau > s)\partial_2 H(s,\xi_s)\Delta X_s\middle| \mathcal{H}_t\right) \\
&- \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}\left(\int_t^\infty \lambda(s)\mathbb{P}(\tau > s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s)+\sigma_{Z,M}(s))\mathrm{d}s\middle| \mathcal{H}_t\right) \quad \text{on } [[0,\tau]]. \qquad (19)
\end{aligned}$$

First of all we note that, by assumption (A3) in Definition 2, and Corollary (2.4) in Revuz and Yor (1999) we have that

$$\lim_{t\uparrow T}\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s)\partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s\middle| \mathcal{H}_t\right) = 0 \quad \text{a.s.}$$

Analogously also $\mathbb{E}\Big(\sum_t^T \mathbb{P}(\tau > s)\partial_2 H(s,\xi_{s-})|\Delta X_s| \Big| \mathcal{H}_t \Big)$ and $\mathbb{E}\Big(\int_t^T \lambda(s)\mathbb{P}(\tau > s)\partial_{22} H(s,\xi_s)(\sigma_M^2(s) + \sigma_{Z,M}(s))\mathrm{d}s \Big| \mathcal{H}_t \Big)$ vanish for $t \uparrow T$. Then, taking the limit in (19), we are left with

$$\lim_{t\uparrow T} \frac{(V_t - H(t,\xi_t))\,\mathbb{P}(\tau > t)}{\lambda(t)} = 0 \quad \text{a.s.} \tag{20}$$

This leads to (i). Moreover, applying the Itô's formula to $\frac{H(t,\xi_t)\mathbb{P}(\tau>t)}{\lambda(t)}$, $t \le \bar{T}$, and studying the limit for $\bar{T} \to T$, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T \mathbb{P}(\tau > s)\,\partial_2 H(s,\xi_{s-})\mathrm{d}X_s \middle| \mathcal{H}_t \right) \\
&= \lim_{\bar{T}\uparrow T} \mathbb{E}\left(\frac{H(\bar{T},\xi_{\bar{T}})\mathbb{P}(\tau > \bar{T})}{\lambda(\bar{T})} \middle| \mathcal{H}_t \right) - \frac{H(t,\xi_t)\mathbb{P}(\tau > t)}{\lambda(t)} \\
&- \mathbb{E}\left(\int_t^T (\partial_s \left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)} \right) H(s,\xi_s) + \frac{\mathbb{P}(\tau > s)}{\lambda(s)} \partial_1 H(s,\xi_s) \right. \\
&\left. + \frac{1}{2}\partial_{22} H(s,\xi_s)\mathbb{P}(\tau > s)\,\lambda(s)\sigma_Y^2(s))\mathrm{d}s \middle| \mathcal{H}_t \right) \\
&- \mathbb{E}\left(\sum_t^T \frac{\mathbb{P}(\tau > s)\Delta H(s,\xi_s)}{\lambda(s)} - \mathbb{P}(\tau > s)\partial_2 H(s,\xi_{s-})\Delta X_s \middle| \mathcal{H}_t \right).
\end{aligned} \tag{21}$$

Moreover, by (20), we have

$$\begin{aligned}
\lim_{\bar{T}\uparrow T} \mathbb{E}\left(\frac{H(\bar{T},\xi_{\bar{T}})\mathbb{P}(\tau > \bar{T})}{\lambda(\bar{T})} \middle| \mathcal{H}_t \right) &= \lim_{\bar{T}\uparrow T} \mathbb{E}\left(\frac{V_{\bar{T}}\mathbb{P}(\tau > \bar{T})}{\lambda(\bar{T})} \middle| \mathcal{H}_t \right) \\
&= V_t \lim_{\bar{T}\uparrow T} \frac{\mathbb{P}(\tau > \bar{T})}{\lambda(\bar{T})} = cV_t.
\end{aligned} \tag{22}$$

By substituting (21) and (22) into (19), we obtain the equation

$$\begin{aligned}
0 &= V_t \left(c - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) - \mathbb{E}\left(\int_t^T \left(\partial_s \left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)} \right) H(s,\xi_s) \right. \right. \\
&\left. \left. + \frac{\mathbb{P}(\tau > s)}{\lambda(s)} \partial_1 H(s,\xi_s) + \frac{1}{2}\partial_{22} H(s,\xi_s)\mathbb{P}(\tau > s)\,\lambda(s)(\sigma_Y^2(s) - 2\sigma_{M,Y}(s)) \right) \mathrm{d}s \middle| \mathcal{H}_t \right) \\
&- \mathbb{E}\left(\sum_t^T \frac{\mathbb{P}(\tau > s)\Delta H(s,\xi_s)}{\lambda(s)} - \mathbb{P}(\tau > s)\partial_2 H(s,\xi_s)\Delta X_s \middle| \mathcal{H}_t \right).
\end{aligned} \tag{23}$$

Proceeding in the same way as in the proof of Proposition 11 with the right-hand side of the equation (9), we can identify the predictive and martingale parts. This yields

$$
\begin{aligned}
0 = & \, \partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) (V_t - H(t, \xi_t)) + \\
& + \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \mathbb{P}(\tau > t) \lambda(t) (\sigma_Y^2(t) - 2\sigma_{M,Y}(t)) \\
& + \left(\frac{\mathbb{P}(\tau > t) \Delta H(t, \xi_t)}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_2 H(t, \xi_t) \Delta \xi_t \right).
\end{aligned} \tag{24}
$$

The above gives us (ii). Now since (H, λ, X) is a local equilibrium, then prices are rational. By taking conditional expectations with respect to $\mathcal{F}_t$, we obtain

$$
\begin{aligned}
0 = & \, \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \mathbb{P}(\tau > t) \lambda(t) \left(\sigma_Y^2(t) - 2\mathbb{E}(\sigma_{M,Y}(t) | \mathcal{F}_t) \right) \\
& + \left(\frac{\mathbb{P}(\tau > t) \Delta H(t, \xi_t)}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_2 H(t, \xi_t) \Delta \xi_t \right).
\end{aligned} \tag{25}
$$

This leads to (iii). ■

The following result is a consequence of the previous result.

Proposition 18 *Consider an admissible triple (H, λ, X) with $\lambda \in C^1$. Moreover, let $H(t, \cdot)$ be linear, for all t, or the strategy X be absolutely continuous. Then if (H, λ, X) is a local equilibrium we have:*

$$(i) \; Y \text{ is an } \mathbb{F}\text{-local martingale} \tag{26}$$

$$(ii) \; \text{If } V_t \neq P_t, \text{ a.s. on } [[0, \tau)), \text{ then } \lambda(t) = c\mathbb{P}(\tau > t) \text{ with } c > 0 \, . \tag{27}$$

Proof. The result derives from the proposition before with an argument similar to the one in Proposition 12. ■

Now we study sufficient conditions for having an equilibrium. We obtain a result in line with Theorem 15.

Theorem 19 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying*

$$\partial_1 H(t, y) + \frac{1}{2} \partial_{22} H(t, y) \lambda^2(t) \sigma^2(t) = 0, \qquad t \geq 0, \tag{28}$$

with σ^2 deterministic and càdlàg and $0 < \sigma^2(t) \leq \sigma_Z^2(t)$, for all $t \geq 0$, with $\partial_{22} H(t, y) > 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$, $\lambda(t) = c\mathbb{P}(\tau > t)$, for all t ($c > 0$), and $\int_0^T \mathbb{E}\left(\left(\partial_2 H(s, \int_0^t \lambda(s)\sigma(s) \mathrm{d}B_s) \right)^2 \right) \lambda^2(s)\sigma^2(s) \mathrm{d}s < \infty$,

where B is a Brownian motion independent of τ. The triple (H, λ, X) is an equilibrium if and only if

$$(i)\ \lim_{\bar{T}\uparrow T} H(\bar{T}, \xi_{\bar{T}}) = \lim_{\bar{T}\uparrow T} V_{\bar{T}},$$

$$(ii)\ \sigma_M^2(t) = \sigma_Z^2(t) - \sigma^2(t),\ 0 \le t \le T,$$

$$(iii)\ Y\ = X + Z \text{ has no jumps,}$$

$$(iv)\ Y_t + \int_0^t \lambda(s) \frac{\partial_{22} H(s, \xi_s)}{\partial_2 H(s, \xi_s)} (\sigma_{M,Z}(s) + \sigma_M^2(s)) \mathrm{d}s,\ 0 \le t \le T, \text{ is an } \mathbb{F}\text{-local martingale.}$$

Proof. See Appendix. ■

For the linear pricing rules case we have a result in line with Theorem 16.

Theorem 20 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (28) with $\partial_{22} H(t, y) = 0$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$ and $\lambda(t) = c\mathbb{P}(\tau > t)$, $c > 0$. Then (H, λ, X) is an equilibrium if and only if the following conditions hold:*

$$(i)\ H(T, \xi_T) = V_T$$

$$(ii)\ \sigma_M(t) = 0,\ 0 \le t \le T,$$

$$(iii)\ Y\ = X + Z \text{ has no jumps}$$

$$(iv)\ Y_t,\ 0 \le t \le T, \text{ is an } \mathbb{F}\text{-local martingale.}$$

Remark 21 *Here we can draw conclusions similar to the one in Cho (2003), where the author considers a risk-averse insider (and a deterministic release time). Cho concludes that, in equilibrium, a risk-adverse insider would do most of her trading early to avoid the risk that the prices get closer to the asset value, unless the trading conditions become more favourable over time. Similarly in our case, when the (risk-neutral) insider does not know the release time of information, she would trade early in order to use her piece of information before the announcement time comes. This behaviour would continue unless the price pressure decreases over time providing more favourable trading conditions also at a later time. A similar conclusion is obtained also by Baruch (2002), who studies the same problem about the effect of risk-aversion for the insider. In his study he assumes that the demand of noise traders follows a Brownian motion with time varying instantaneous variance.*

Example 22 *We can consider the context of Caldentey and Stacchetti (2010) where the authors assume that V and Z are arithmetic Brownian motion with variances σ_V and σ_Z, respectively, and that τ follows an*

exponential distribution with scale parameter μ, independent of (V, P, Z). Then, by Proposition 18, we have that, for a.a. t and a.a. $\omega \in \{t < \tau\}$,

$$V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}\left(\int_t^{\infty} e^{-\mu(s-t)} \partial_2 H(s, \xi_s) \mathrm{d}X_s \middle| \mathcal{H}_t \right) = 0.$$

And to have a local equilibrium, provided that $V_t - H(t, \xi_t) \neq 0$, we need $\lambda(t) = \lambda_0 e^{-\mu t}$.

5 Explicit insider's optimal strategies

In this section we shall apply our results to explicitly find the insider's optimal strategy in equilibrium. We will show how our general framework serves different models known in the literature, which are presented as various different extensions of the Kyle-Back model. In order to perform the explicit computations we will use techniques both of enlargements of filtrations and of filtering. We shall illustrate them with several pilot examples.

To explain how enlargement of filtration enters the topic we consider a total demand $Y = Z + X$ in equilibrium given by:

$$Y_t = Z_t + \int_0^t \theta(\eta_u, Y_u, 0 \leq u \leq s) \mathrm{d}s, \quad 0 \leq t \leq T. \tag{29}$$

Here X is an absolutely continuous process with respect to the Lebesgue measure. We recall that Z is perceived by the insider as an $\mathbb{H}$-martingale independent of V and η. Let $\mathbb{F}^{Y,\eta} = (\mathcal{F}_t^{Y,\eta})_{t \geq 0}$ be the filtration $\mathcal{F}_t^{Y,\eta} := \bar{\sigma}(Y_s, \eta_s, 0 \leq s \leq t)$. Since $\mathcal{F}_t^{Y,\eta} \subseteq \mathcal{H}_t$, for all t, and Z is adapted to $\mathbb{F}^{Y,\eta}$, we see that Z is also an $\mathbb{F}^{Y,\eta}$-martingale. On the other hand Y is, in certain cases as in Theorem 15, Proposition 11 and Proposition 18, a local martingale when in equilibrium. Consequently (29) becomes the Doob-Meyer decomposition of Y when we enlarge the filtration $\mathbb{F}$ with the process η. We are then into a problem of *enlargement of filtrations.* However, in our problem Z is fixed in advance and we want to obtain Y as a function of Z, given η, so we look in fact for *strong* solutions of (29), whereas the results on enlargement of filtrations provide weak solutions. Then we can call upon the Yamada-Watanabe's theorem, when Z is Gaussian, to obtain strong solutions from weak solutions. See, for instance, Theorem 1.5.4.4. in Jeanblanc *et al.* (2009).

The following various examples correspond to different models, which are all extensions of the Kyle-Back model and where the results about enlargement of filtrations can be applied. We will not enter, however, into the details on the derivation of a strong solution in the corresponding stochastic differential equations appearing in equilibrium.

Example 23 *(Aase et al. (2012a)) Assume that $\tau = 1$ and suppose that Z is given by*

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic. In equilibrium, if the strategy of the insider is optimal $V_1 = H(1, Y_1)$. Since $H(1, \cdot)$ can be chosen freely because it is the boundary condition of equation (14) and if V_1 has a continuous cumulative distribution function, we can assume without loss of generality that $Y_1 \equiv N(0, \int_0^1 \sigma_s^2 \mathrm{d}s)$. It is assumed that V_1 (and consequently Y_1) is independent of Z. Then by Jeulin (1980), page 51,

$$Y_t = Z_t + \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s,$$

has the same law as Z. Then

$$X_t = \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s$$

is the optimal strategy. As a particular case we find the study of Back (1992) where $\sigma_s^2 \equiv \sigma^2$.

Example 24 *(Campi and Çetin (2007)) If we want both the total aggregate demand process Y to be a Brownian motion that reaches the value -1 for the first time at time $\bar{\tau}$, and the aggregate demand of the liquidity traders Z to be also a Brownian motion, then by Example 3 in Jeulin and Yor (1985), page 306, it is*

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1 + Y_s} - \frac{1 + Y_s}{\bar{\tau} - s} \right) \mathbf{1}_{[0, \bar{\tau}]}(s) \mathrm{d}s.$$

So, in this case, we can refer to our framework by taking $\eta_t \equiv \bar{\tau}$, $V_t \equiv \mathbf{1}_{\{\bar{\tau} > 1\}}$ and the release time $\tau = \bar{\tau} \wedge 1$.

Another point of view of the problem of finding the equilibrium strategy is the following. Market makers observe Y with dynamics

$$\mathrm{d}Y_t = \mathrm{d}Z_t + \theta(V_s, Y_s, 0 \leq s \leq t)\mathrm{d}t,$$

while V is not observed. Then, the dynamics of $m_t := \mathbb{E}(V_t | \mathcal{F}_t)$ can be obtained in certain cases (basically when Z and V are Gaussian diffusions) from the filtering theory, see for instance Theorem 12.1 in Liptser and Shiryaev (1978). Now we can try to deduce $\theta(V_s, Y_s, 0 \leq s \leq t)$ from the equilibrium condition: $P_t = m_t$. Note that even if V is not a Gaussian diffusion, but can be written in the form $V_t = h(D_t)$ where h is a strictly increasing function and D is a Gaussian diffusion, we can still apply the filtering results for the couple (Y, D).

In the following example we use the filtering approach to find the equilibrium strategy.

Example 25 *(Caldentey and Stacchetti (2010)) The context is as follows. The release time τ is unknown and*

$$\mathrm{d}V_t = \sigma_v(t)\mathrm{d}B_t^v, \qquad V_0 \sim N(P_0, \Sigma_0)$$
$$\mathrm{d}Z_t = \sigma_z(t)\mathrm{d}B_t^z, \qquad Z_0 = 0,$$

where B^v and B^z are independent Brownian motions and $\sigma_v(t)$ and $\sigma_z(t)$ are deterministic functions. If we look for pricing rules such that

$$\mathrm{d}P_t = \lambda(t)\mathrm{d}Y_t$$

and strategies with form

$$\mathrm{d}X_t = \beta(t)(V_t - P_t)\mathrm{d}t,$$

with $\beta(t)$ deterministic, we have that

$$\mathrm{d}P_t = \lambda(t)\beta(t)(V_t - P_t)\mathrm{d}t + \lambda(t)\sigma_z(t)\mathrm{d}B_t^z.$$

Let $m_t := E(V_t|\mathcal{F}_t^Y)$. By standard filtering results (see for instance Lipster and Shiryayev (2001)) we have

$$\mathrm{d}m_t = \frac{\Sigma_t\beta(t)}{\lambda(t)\sigma_z^2(t)}(\mathrm{d}P_t - \lambda(t)\beta(t)(m_t - P_t)\mathrm{d}t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = \sigma_v^2(t) - \frac{(\Sigma_t\beta(t))^2}{\sigma_z^2(t)},$$

where Σ_t is the filtering error. Now, we can recover the identity $P_t = m_t$ if and only if we impose $\Sigma_t\beta(t) = \lambda(t)\sigma_z^2(t)$ (remember that by construction $P_0 = m_0 = E(V_0)$) . Then

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \int_0^t \sigma_z^2(s)\lambda^2(s)\mathrm{d}s, \ \beta(t) = \frac{\lambda(t)\sigma_z^2(t)}{\Sigma_t}.$$

Note that in particular we obtain that

$$Y_t = Z_t + \int_0^t \frac{\lambda(s)(\sigma_z^2(s)\left(V_s - \int_0^s \lambda(u)\mathrm{d}Y_u\right)}{\Sigma_s}\mathrm{d}s,$$

is the Doob-Meyer decomposition of the martingale Y in the filtration generated by (Z, V). Now if we assume $\sigma_z^2(t) = \sigma_z^2$, independent of t, and we take into account that in the equilibrium $\lambda(t) = \lambda_0 e^{-\mu t}$, we have that

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \sigma_z^2\frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu t}), \beta(t) = \frac{\sigma_z^2\lambda_0 e^{-\mu t}}{\Sigma_t}.$$

However λ_0 is not determined. We need an additional condition to fix λ_0. One possibility is to impose that

$$\lim_{t\to\infty} \Sigma_t = 0.$$

In such a case

$$0 = \Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s - \sigma_z^2 \frac{\lambda_0^2}{2\mu},$$

and

$$\lambda_0 = \sqrt{\frac{2\mu(\Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s)}{\sigma_z^2}}.$$

Note that if $\sigma_v^2(t) = \sigma_v^2$ there is no solution!

Example 26 *(Caldentey and Stacchetti (2010) continued) Hereafter we can discuss other types of strategies X in the same context of Example 25. For instance we can consider strategies involving a time T representing the time when the insider releases all the information to the market. With this kind of strategies, according with Proposition 18, the time T is such that the filtering error is*

$$\Sigma_t = 0, \qquad \textit{for all } t \geq T.$$

Then $P_t = V_t$ for $t \geq T$. But this implies, for $\sigma_v^2(t) = \sigma_v^2$,

$$\begin{aligned} 0 &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu T}) \\ &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_T^2}{2\mu}(e^{2\mu T} - 1). \end{aligned}$$

Now if we assume a smooth transition from the absolutely continuous strategy to the unbounded variation one, that is $\dot{\Sigma}_t = 0$, for all $t \geq T$, then $\sigma_v^2 - \sigma_z^2\lambda^2(t) = 0$ and $\lambda(t) = \lambda_T = \frac{\sigma_v}{\sigma_z}$ for all $t \geq T$. Finally

$$\mathrm{d}P_t = \lambda(t)\mathrm{d}Y_t = \lambda(t)\mathrm{d}X_t + \lambda(t)\mathrm{d}Z_t = \mathrm{d}V_t, \quad t \geq T$$

so

$$\mathrm{d}X_t = \frac{\sigma_z}{\sigma_v}\mathrm{d}V_t - \mathrm{d}Z_t,$$

and T is the solution of

$$\Sigma_0 + \sigma_v^2 T = \frac{\sigma_v^2}{2\mu}(e^{2\mu T} - 1).$$

This is exactly what Caldentey and Stacchetti (2010) obtain. It is important to remark that the authors obtain

a limit of optimal strategies when passing from the discrete time version of the model to the continuous one. This limit strategy is such that there is an endogenously determined time T such that, if $t \leq T$, then the limit strategy is absolutely continuous with respect to the Lebesgue measure and, if $t > T$, the strategy is not of bounded variation. In this case an insider's optimal strategy, between times T and τ, would yield to giving out the full information to the market by making the market prices match the fundamental value. The authors claim that this limit strategy is not optimal for the continuous time model and that we need to consider the discrete time model to realize about its existence. With respect to this point we remark that this limit strategy can be obtained as a limit of strategies for the continuous time model when we restrict the class of strategies to the set of those absolutely continuous and then we maximize the wealth. In fact, if we have a sequence of strategies $\left(X^{(n)}\right)_{n\geq 1}$, their corresponding wealth is given by

$$W_\tau^{(n)} = X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} - [P^{(n)}, X^{(n)}]_\tau.$$

Then, if we assume that $(X^{(n)}, P^{(n)}, V^{(n)}) \underset{n\to\infty}{\overset{u.c.p}{\to}} (X, P, V)$ *we obtain that*

$$X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} \underset{n\to\infty}{\overset{u.c.p}{\to}} X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t$$

but, in general,

$$[P^{(n)}, X^{(n)}]_\tau \nrightarrow [P, X]_\tau.$$

For instance, if $X^{(n)}$ is a bounded variation process, then X is not necessarily a bounded variation one. Then the gain for this limit of strategies after T, on the set $\{\tau > T\}$, is given by

$$\begin{aligned} V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-} \mathrm{d}X_t &= \int_T^\tau X_{t-} \mathrm{d}V_t + \int_T^\tau V_{t-} \mathrm{d}X_t + \int_T^\tau \mathrm{d}[V, X]_t - \int_T^\tau P_{t-} \mathrm{d}X_t \\ &= \int_T^\tau (V_{t-} - P_{t-})\, \mathrm{d}X_t + \int_T^\tau \mathrm{d}[V, X]_t + \int_T^\tau X_{t-} \mathrm{d}V_t. \end{aligned}$$

Now if we take the conditional expectation, last term of the right-hand side cancels and we obtain that

$$\mathbb{E}\left(V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-} \mathrm{d}X_t \middle| \mathcal{H}_T \right) = \mathbb{E}\left(\int_T^\tau (V_{t-} - P_{t-})\, \mathrm{d}X_t + \int_T^\tau \mathrm{d}[V, X]_t \middle| \mathcal{H}_T \right).$$

Finally, since $V_{t-} = P_{t-}$, $t > T$ for the limit strategy, in the conditions of Example 22, we obtain that there is a profit after T given by

$$\mathbb{E}\left(\int_T^\infty e^{-\mu(t-T)} \mathrm{d}[V, X]_t \middle| \mathcal{H}_T \right) = \sigma_z \sigma_v \int_T^\infty e^{-\mu(t-T)} \mathrm{d}t = \frac{\sigma_z \sigma_v}{\mu} > 0.$$

Now we can justify the condition $\dot{\Sigma}_T = 0$. The expected wealth for the insider with this kind of strategies is given by

$$
\begin{aligned}
J(X) &= \mathbb{E}\left(\int_0^{T\wedge\tau}(V_t - P_t)\theta_t \mathrm{d}t\right) + \mathbb{E}\left(\int_{T\wedge\tau}^{\tau}\mathrm{d}[V,X]_t\right) = \mathbb{E}\left(\int_0^{T\wedge\tau}\beta_t(V_t - P_t)^2\mathrm{d}t\right) + \mathbb{E}\left(\int_{T\wedge\tau}^{\tau}\mathrm{d}[V,X]_t\right) \\
&= \mathbb{E}\left(\int_0^{T}\mathbf{1}_{[0,\tau]}(t)\beta_t(V_t - P_t)^2\mathrm{d}t\right) + \mathbb{E}\left(\int_{T}^{\infty}\mathbf{1}_{[0,\tau]}(t)\mathrm{d}[V,X]_t\right) = \int_0^T \mathbb{P}(\tau > t)\beta_t\Sigma_t\mathrm{d}t + \int_T^\infty \mathbb{P}(\tau > t)\frac{\sigma_v^2}{\lambda_t}\mathrm{d}t \\
&= \int_0^T e^{-\mu t}\beta_t\Sigma_t\mathrm{d}t + \sigma_v^2\int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t = \sigma_z^2\int_0^T e^{-\mu t}\lambda_t\mathrm{d}t + \sigma_v^2\int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t.
\end{aligned}
$$

Then if we impose that T is optimal, we have the condition

$$\sigma_z^2 e^{-\mu T}\lambda_T - \sigma_v^2\frac{e^{-\mu T}}{\lambda_T} = 0,$$

that is

$$\lambda_T = \frac{\sigma_v}{\sigma_z},$$

and this is equivalent to $\dot{\Sigma}_T = 0$. Note that other equilibria are possible by taking $\lambda_t \neq \lambda_T$ when $t > T$.

Remark 27 *It can be proved that the linearity of the strategies assumed in the previous example implies that the equilibrium pricing rules have to be linear as well. This interesting result can be seen also in Aase et al. (2012a).*

Example 28 *Another interesting example is that of Campi et al. (2013). There, authors consider a defaultable stock. The default time is modeled as the first time that a Brownian motion, say B, hits the barrier -1, as in the above Example 24 . However in this case the default time, $\tau = \inf\{t \geq 0, B_t = -1\}$, is not known by the insider, but it is a stopping time for every trader. Instead, the insider observes the process $\left(B_{r(t)}\right)$ where $r(t)$ is a deterministic, increasing function with $r(t) > t$ for $t \in (0,1)$, $r(0) = 0$, and $r(1) = 1$. This circumstance allows the insider to know in advance the default time. The horizon of the market is $t = 1$. The authors also consider a payoff of the kind $f(B_1)$ in case of no default. Note that $\tau = r(\delta \wedge 1)$, where $\delta = \inf\{0 \leq t \leq 1, B_{r(t)} = -1\}$ and $\delta = \infty$ if the previous set is empty. Then, in this example, treated in our framework, the release time is $\tau = r(\delta \wedge 1)$, the signal is $\eta_t = B_{r(t)}$ and the fundamental value is*

$$V_t = \mathbb{E}(f(B_1)\mathbf{1}_{\{\delta>\tau\}}|B_{r(t)}).$$

Moreover the aggregate demand of noise traders Z follows a Brownian motion, say W, so $Z = W$. Even though τ (and consequently δ) is not known by the insider, it is a predictable stopping time. We can see

that the price pressure λ is constant by an extension of the case considered in Section 3 and that the optimal strategy moves prices to the fundamental one:

$$\lim_{\delta_n \uparrow \delta} P_{\delta_n} = V_\delta,$$

where (δ_n) is any increasing sequence of stopping times that grows to δ. To find the explicit form of an equilibrium strategy is not straightforward. However, if $\tau \leq s \leq V(\tau)$, an equilibrium strategy is obtained from a strong solution of

$$Y_s = W_s + \int_0^s \left(\frac{1}{1+Y_u} - \frac{1+Y_u}{V(\tau)-u} \right)(u)\mathrm{d}u,$$

as we deduce from Example 24 above. The difficult part is to see what happens for $s < \tau$. This requires a quite involved use of enlargement of filtrations and filtering techniques. See Campi et al. (2013b) for the details.

6 Appendix

Proof. (Theorem 19) Assume $(i)-(iv)$ hold true. We show that (H, λ, X) is an equilibrium. Define

$$\varsigma_t := \int_0^t \lambda(s)\sigma(s)\mathrm{d}B_s.$$

where B is a Brownian motion independent of τ. First if $H(t,y)$ is a solution of (28)

$$H(t, \varsigma_t) = H(0,0) + \int_0^t \partial_2 H(s,\varsigma_s)\lambda(s)\sigma(s)\mathrm{d}B_s$$

then, since $\int_0^T \mathbb{E}\left((\partial_2 H(s,\varsigma_s))^2\right)\lambda^2(s)\sigma^2(s)\mathrm{d}s < \infty$, $(H(t,\varsigma_t))_{t\geq 0}$ is a martingale (w.r.t. its own filtration). Then for $T \in [0,\infty]$ and $t < T$

$$H(t,y) = \mathbb{E}\left(H(T,\varsigma_T) \middle| \varsigma_t = y\right) = \mathbb{E}\left(H(T, y + \varsigma_T - \varsigma_t)\right)$$

(This is well defined by condition $\int_0^T \mathbb{E}\left((\partial_2 H(s,\varsigma_s))^2\right)\lambda^2(s)\sigma^2(s)\mathrm{d}s < \infty$, $\lim_{\bar{T}\uparrow T} H\left(\bar{T}, \varsigma_{\bar{T}}\right) = H(T,\varsigma_T)$ in L^2). Set now,

$$i(T,y,v) := \int_y^{H^{-1}(T,\cdot)(v)} \frac{v - H(T,x)}{c}\mathrm{d}x$$

with $c = \frac{\lambda(t)}{\mathbb{P}(\tau > t)}$and $H^{-1}(T, \cdot)(v) := \lim_{\bar{T}\uparrow T} H^{-1}(\bar{T}, \cdot)(v)$. Define

$$I(t, y, v) := \mathbb{E}\left(i\left(T, y + \varsigma_T - \varsigma_t, v\right)\right), \quad t \geq 0,$$

we have that

$$\begin{aligned} \partial_2 I(t, y, v) &= \mathbb{E}\left(\partial_1 i(T, y + \varsigma_T - \varsigma_t, v)\right) \\ &= \mathbb{E}\left(-\frac{v - H(T, y + \varsigma_T - \varsigma_t)}{c}\right) = -\frac{v - H(t, y)}{c}. \end{aligned} \tag{30}$$

We can take the derivative under the integral sign because $H(T, \cdot)$ is monotone and $\mathbb{E}\left(H(T, \varsigma_T)\right) < \infty$. Then $I(t, y, v)$ is well defined and

$$\begin{aligned} I(t, y, v) &= \mathbb{E}\left(i\left(T, y + \varsigma_T - \varsigma_t, v\right)\right) \\ &= \mathbb{E}\left(i\left(T, \varsigma_T, v\right) \middle| \varsigma_t = y\right), \end{aligned}$$

and $\left(I\left(t, \varsigma_t, v\right)\right)_{t \geq 0}$ is a martingale (w.r.t. its own filtration), so

$$\partial_1 I\left(t, \varsigma_t, v\right) + \frac{1}{2} \partial_{22} I\left(t, \varsigma_t, v\right) \lambda(t)^2 \sigma^2(t) = 0. \tag{31}$$

Now, consider any admissible strategy X, by using Itô's formula, we have

$$\begin{aligned} I(T, \xi_T, V_T) = I(0, 0, V_0) &+ \int_0^T \partial_3 I(t, \xi_t, V_t) \mathrm{d}V_t + \int_0^T \partial_1 I(t, \xi_t, V_t) \mathrm{d}t \\ &+ \int_0^T \partial_2 I(t, \xi_{t-}, V_t) \mathrm{d}\xi_t + \frac{1}{2} \int_0^T \partial_{22} I(t, \xi_t, V_t) \mathrm{d}[\xi^c, \xi^c]_t \\ &+ \int_0^T \partial_{23} I(t, \xi_t, V_t) \mathrm{d}[\xi^c, V]_t + \frac{1}{2} \int_0^T \partial_{33} I(t, \xi_t, V_t) \sigma_V^2(t) \mathrm{d}t \\ &+ \sum_{0 \leq t < T} \left(\Delta I(t, \xi_t, V_t) - \partial_2 I(t, \xi_{t-}, V_t) \Delta \xi_t\right). \end{aligned}$$

By construction, $\xi_0 = 0$, by (i) $\mathrm{d}\xi_t = \lambda(t) \mathrm{d}Y_t$. Now we have that

$$\mathrm{d}[\xi^c, \xi^c]_t = \lambda(t)^2 \mathrm{d}[X^c, X^c]_t + 2\lambda(t)^2 \mathrm{d}[X^c, Z]_t + \lambda(t)^2 \sigma_Z^2(t) \mathrm{d}t.$$

Also by (30) and the fact that V and Z are independent,

$$\partial_{23} I(t, \xi_t, V_t) \mathrm{d}[\xi^c, V]_t = -\frac{1}{c} \mathrm{d}[\xi^c, V]_t = -\mathbb{P}(\tau > t) \mathrm{d}[X, V]_t,$$

then using (30) and (31), and the fact that Z has not jumps, we get

$$
\begin{aligned}
I(T,\xi_T,V_T) = I(0,0,V_0) &+ \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau>t)(P_{t-}-V_t)(\mathrm{d}X_t+\mathrm{d}Z_t)\\
&+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,X^c]_t - \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[X,V]_t + \frac{1}{2}\int_0^\tau \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t\\
&+ \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,Z] + \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t)-\sigma^2(t)\right)\mathrm{d}t\\
&+ \sum_{0\le t\le\tau} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\lambda(t)\Delta X_t\right).
\end{aligned}
$$

Subtracting $\int_0^T \mathbb{P}(\tau>t)\mathrm{d}[P,X]_t$ from both sides and rearranging the terms, we obtain

$$
\begin{aligned}
&\int_0^T \mathbb{P}(\tau>t)(V_t-P_{t-})\mathrm{d}X_t - \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[P,X]_t + \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[X,V]_t\\
&- \left(I(0,0,V_0) + \frac{1}{2}\int_0^T \partial_{33} I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t\right)\\
&= -I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_{t-},V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau>t)(P_t-V_t)\mathrm{d}Z_t\\
&+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,X^c]_t + \int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}[X^c,Z]_t\\
&+ \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t)-\sigma^2(t)\right)\mathrm{d}t\\
&+ \sum_{0\le t<T} \left(\Delta I(t,\xi_t,V_t) - \partial_2 I(t,\xi_{t-},V_t)\lambda(t)\Delta X_t\right) - \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[P,X]_t. \qquad (32)
\end{aligned}
$$

We have that

$$
\mathbb{P}(\tau>t)\mathrm{d}[P,X]_t = \mathbb{P}(\tau>t)\mathrm{d}[P^c,X^c]_t + \mathbb{P}(\tau>t)\Delta P_t\Delta X_t.
$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c$, so by using (30), we obtain

$$
\begin{aligned}
\mathbb{P}(\tau>t)\mathrm{d}[P^c,X^c]_t &= \frac{\lambda(t)}{c}\partial_2 H(t,\xi_t)\mathrm{d}\left[\xi^c,X^c\right]_t\\
&= \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}\left[X^c,X^c\right]_t + \int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\mathrm{d}\left[X^c,Z\right]_t,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda(t)\partial_2 I(t,\xi_{t-},V_t)\Delta X_t + \mathbb{P}(\tau>t)\Delta P_t\Delta X_t &= \frac{\lambda(t)}{c}(P_{t-}-V_t)\Delta X_t + \frac{\lambda(t)}{c}\Delta P_t\Delta X_t\\
&= \frac{\lambda(t)}{c}(P_t-V_t)\Delta X_t = \lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t.
\end{aligned}
$$

Substituting the above relationships in the right-hand side of the equation (32), it becomes

$$
\begin{aligned}
&- I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau>t)(P_t-V_t)\mathrm{d}Z_t - \frac{1}{2}\int_0^T \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2 \mathrm{d}[X^c,X^c]_t \\
&+ \frac{1}{2}\int_0^\tau \partial_{22} I(t,\xi_t,V_t)\lambda(t)^2\left(\sigma_Z^2(t)-\sigma^2(t)\right)\mathrm{d}t \\
&+ \sum_{0\le t\le \tau}\left(I(t,\xi_t,V_t)-I(t,\xi_{t-},V_t)-\lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t\right) \\
&= -I(T,\xi_T,V_T) + \int_0^T \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t + \int_0^T \mathbb{P}(\tau>t)(P_t-V_t)\mathrm{d}Z_t \\
&+ \sum_{0\le t< T}\left(I(t,\xi_t,V_t)-I(t,\xi_{t-},V_t)-\lambda(t)\partial_2 I(t,\xi_t,V_t)\Delta X_t\right).
\end{aligned}
$$

Now, observe that $\partial_{33}I(t,y,v)$ does not depend on y and so $\partial_{33}I(t,\xi_t,V_t)$ does not depend of ξ. Then $I(0,0,V_0)+\frac{1}{2}\int_0^T \partial_{33}I(t,\xi_t,V_t)\sigma_V^2(t)\mathrm{d}t$ has values that does not depend on the strategy. Then on the left-hand side of (32) only the term

$$\int_0^T \mathbb{P}(\tau>t)(V_t-P_{t-})\mathrm{d}X_t - \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[P,X]_t + \int_0^T \mathbb{P}(\tau>t)\mathrm{d}[X,V]_t$$

depends on the strategy and its expectation is just the value of (4). Then we show that, taken the expectation, the right-hand side of (32) achieves its maximum value at X. The result follows from the following points.

1. (iii) guarantees that $\Delta X_t = 0$
2. The processes $\int_0^\cdot \partial_3 I(t,\xi_t,V_t)\mathrm{d}V_t$ and $\int_0^\cdot (P_t-V_t)\mathrm{d}Z_t$ are $\mathbb{H}$-martingales by (A5) and (A2) in Definition 2, hence they have null expectation
3. We know that $c\partial_{22}I(\bar{T},\xi_{\bar{T}},V_{\bar{T}}) = \partial_2 H(\bar{T},\xi_{\bar{T}})>0$ and that $c\partial_2 I(\bar{T},\xi_{\bar{T}},V_{\bar{T}}) = -V_{\bar{T}}+H(\bar{T},\xi_{\bar{T}})$ so by (i) we have a maximum value of $-I(T,\xi_T,V_T)$ for our strategy.

Assumption (iv) and (i) together with condition (A2) in Definition 2 guarantee the rationality of prices, given X. In fact from (28)

$$\mathrm{d}P_t = \lambda(t)\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \frac{1}{2}\lambda(t)^2\left(\sigma_Y^2(t)-\sigma^2(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t$$

and by (ii)

$$
\begin{aligned}
\mathrm{d}P_t &= \lambda(t)\partial_2 H(t,\xi_t)\mathrm{d}Y_t + \lambda(t)^2\left(\sigma_M^2(t)+\sigma_{M,Z}(t)\right)\partial_{22}H(t,\xi_t)\mathrm{d}t \\
&= \lambda(t)\partial_2 H(t,\xi_t)\left(\mathrm{d}Y_t + \lambda(t)\left(\sigma_M^2(t)+\sigma_{M,Z}(t)\right)\frac{\partial_{22}H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t\right)
\end{aligned}
$$

so, P is an $\mathbb{F}$-local martingale and, by condition (A2) in Definition 2, it is an $\mathbb{F}$-martingale. Then from (i), and on the set $\{t \leq \tau\}$ we have

$$\mathbb{E}\left(H(T,\xi_T) \middle| \mathcal{F}_t\right) = \mathbb{E}\left(V_T \middle| \mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(V_T \middle| \mathcal{H}_t\right) \middle| \mathcal{F}_t\right) = \mathbb{E}\left(V_t \middle| \mathcal{F}_t\right).$$

Conversely, assume that (H, λ, X) is an equilibrium. We show that $(i) - (iv)$ hold true. First note that (i) is a necessary condition for equilibrium by (i) in Proposition 17. Also (ii) is a necessary condition, in fact by (ii) in Proposition 17 and since $\lambda(t) = c\mathbb{P}(\tau > t)$ we obtain that, in the equilibrium,

$$\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(s)^2(\sigma_Z^2(t) - \sigma_M^2(t)) = 0.$$

Since $\partial_{22} H(t, y) > 0$ and (28) holds, we have that $\sigma_Z^2(t) - \sigma_M^2(t) = \sigma^2(t)$. From the computations above we can see that $\partial_{22} I = \frac{\partial_2 H}{c} > 0$ (convexity) implies that

$$I(t, x+h, v) - I(t, x, v) - \partial_2 I(t, x, v)h \leq 0, \quad \text{for any } h.$$

So,

$$\sum_{0 \leq t \leq \tau} \left(I(t, \xi_{t-} + \lambda(t)\Delta X_t, V_t) - I(t, \xi_{t-}, V_t) - \partial_2 I(t, \xi_t, V_t)\lambda(t)\Delta X_t\right) \leq 0,$$

and has its maximum at X. So (iii) is a necessary condition for equilibrium. Finally, we have that

$$\mathrm{d}Y_t + \lambda(t)\left(\sigma_M^2(t) + \sigma_{M,Z}(t)\right)\frac{\partial_{22} H(t,\xi_t)}{\partial_2 H(t,\xi_t)}\mathrm{d}t = \frac{\mathrm{d}P_t}{\lambda(t)\partial_2 H(t,\xi_t)}$$

and since prices are rational, given X, we can see that (iv) holds. ■

Acknowledgement. We would like to thank José Fajardo for helpful discussions and advice and also anonymous reviewers of an earlier version of this work for their comments. This research was partially carried out at CAS - Centre for Advanced Study at the Norwegian Academy of Science and Letters, Research group SEFE.

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